Pismeni ispit iz Analize III, 01.07.2013.
ispit pisati isključivo hemiskom olovkom

1. (*a*) Ako je $z = \ln(e^x + e^t)$ gdje je $x = t^3$ izračunati $\dfrac{\partial z}{\partial t}$ i $\dfrac{\mathrm{d}z}{\mathrm{d}t}$.

(*b*) Provjeriti da li funkcija $u = \sin x + F(\sin y - \sin x)$, u kojoj je F diferencijabilna funkcija, zadovoljava jednakost $\dfrac{\partial u}{\partial y}\cos x + \dfrac{\partial u}{\partial x}\cos y = \cos x \cos y$.

2. Izračunati $I = \iint\limits_G \left(x + \dfrac{y^2}{x^2}\right) \mathrm{d}x\mathrm{d}y$ gdje je $G = \{(x,y) : x^2 + y^2 - 2ax \leq 0, a > 0\}$.

3. Odrediti površinu koju cilindar $x^2 + y^2 = ax$ isjeca na lopti $x^2 + y^2 + z^2 = a^2$ iznad ravni Oxy.

4. Pokazati da je vektorsko polje $\vec{v} = (2x + y + z, x + 2y + z, x + y + z)$ potencijalno i naći njegov potencijal.

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Za uočene greške pisati na *infoarrt@gmail.com*

(#) Ako je $z = \ln(e^x + e^t)$ gdje je $x = t^3$

izračunati $\frac{\partial z}{\partial t}$ i $\frac{dz}{dt}$.

Rj:

$\frac{\partial z}{\partial t}$ je parcijalni izvod po t-u

$$\frac{\partial z}{\partial t} = \frac{1}{e^x + e^t} \cdot e^t = \frac{e^t}{e^x + e^t}$$

$\frac{dz}{dt}$ je izvod po t složene f-je z (čija je promjenljiva t)

Izvod ćemo naći po formuli

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial t}$$

$$\frac{\partial z}{\partial x} = \frac{e^x}{e^x + e^t}, \quad \frac{dx}{dt} = 3t^2$$

$$\frac{dz}{dt} = \frac{3t^2 e^x}{e^x + e^t} + \frac{e^t}{e^x + e^t} = \frac{3t^2 e^x + e^t}{e^x + e^t}$$

Provjeriti da li f-ja $u = \sin x + F(\sin y - \sin x)$ u kojoj je F diferencijabilna f-ja zadovoljava jednakost

$$\frac{\partial u}{\partial y}\cos x + \frac{\partial u}{\partial x}\cos y = \cos x \cos y.$$

Rj: Označimo sa $v = \sin y - \sin x$. Imamo

$$u = \sin x + F(v)$$

u je složena f-ja promjenjivih x i y.
Koristimo formulu

$$\frac{\partial u}{\partial x} = \frac{\partial \sin x}{\partial x} + \frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial \sin x}{\partial y} + \frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial y}$$

Imamo:

$$\frac{\partial u}{\partial x} = \cos x + F'_v\cdot(-\cos x) \Rightarrow \frac{\partial u}{\partial x}\cdot\cos y = \cos x\cos y - \cos x \cos y\, F'_v$$

$$\frac{\partial u}{\partial y} = 0 + F'_v\cdot\cos y \Rightarrow \frac{\partial u}{\partial y}\cdot\cos x = \cos x\cos y\, F'_v \quad +$$

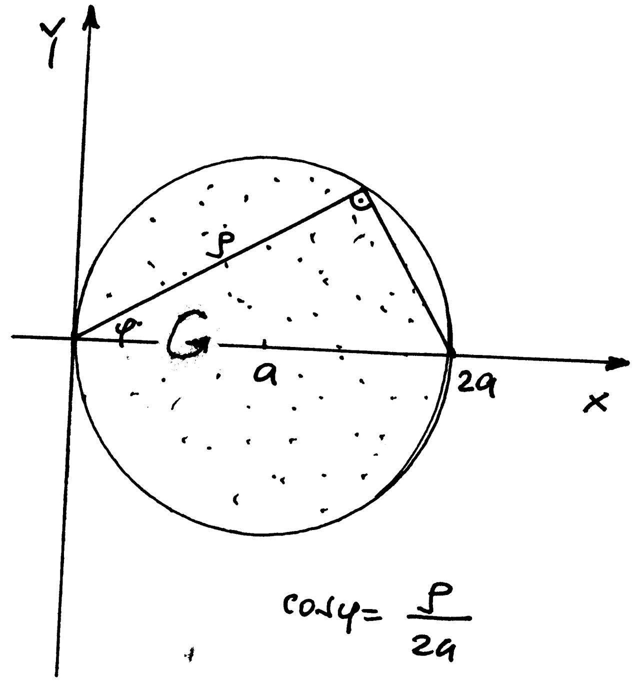
$$\frac{\partial u}{\partial y}\cos x + \frac{\partial u}{\partial x}\cos y = \cos x\cos y$$

vrijedi data jednakost

(#) Izračunati $I=\iint\limits_G \left(x+\frac{y^2}{x^2}\right)dx\,dy$ gdje je

$G=\{(x,y):\ x^2+y^2-2ax\le 0,\ a>0\}$.

Rj. Skicirajmo oblast G



$x^2+y^2-2ax=0$

$x^2-2\cdot x\cdot a+a^2-a^2+y^2=0$

$(x-a)^2+y^2=a^2$

krug sa centrom u tački $C(a,0)$ poluprečnika $r=a$

$\cos\varphi=\frac{\rho}{2a}$

$\rho=2a\cos\varphi$

Uvedimo polarne koordinate

$x=\rho\cos\varphi$

$y=\rho\sin\varphi$

$dx\,dy=\rho\,d\rho\,d\varphi$

$G \xrightarrow{\text{transformiše}} G':\begin{cases}-\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}\\ 0\le\rho\le 2a\cos\varphi\end{cases}$

$$I=\iint\limits_G\left(x+\frac{y^2}{x^2}\right)dx\,dy=\left|\begin{matrix}\text{uvodimo}\\ \text{polarne}\\ \text{koordinate}\end{matrix}\right|=\iint\limits_{G'}\left(\rho\cos\varphi+\frac{\sin^2\varphi}{\cos^2\varphi}\right)\rho\,d\rho\,d\varphi$$

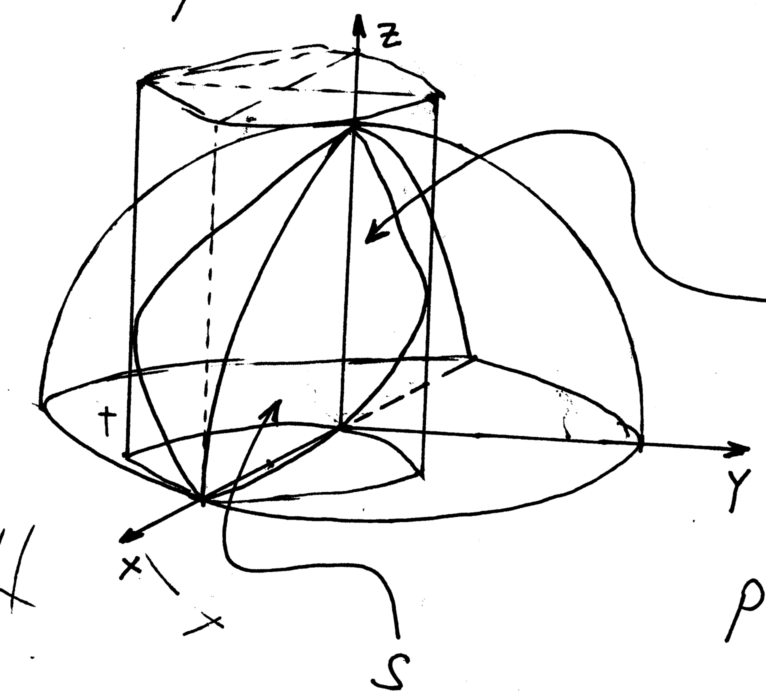
$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\varphi\,d\varphi\int_0^{2a\cos\varphi}\rho^2\,d\rho+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin^2\varphi}{\cos^2\varphi}\,d\varphi\int_0^{2a\cos\varphi}\rho\,d\rho=\dots=\frac{8a^3}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\varphi\,d\varphi+$$

$$+2a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin^2\varphi}{\cos^2\varphi}\cdot\cos^2\varphi\,d\varphi=\dots=a^3\pi+a^2\pi=a^2(a+1)\pi$$ traženo rješenje

Odrediti površinu koju cilindar $x^2+y^2=ax$ isjeca na lopti $x^2+y^2+z^2=a^2$ iznad ravni Oxy.

Rj.

Skiciramo sliku



$$x^2+y^2=ax$$

$$x^2-2\cdot x\cdot\frac{a}{2}+\frac{a^2}{4}-\frac{a^2}{4}+y^2=0$$

$$\left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2$$

Trebamo izračunati površinu dijela lopte koji se nalazi unutar cilindra.

$$P=\iint\limits_S dS=\iint\limits_D\sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$$

$$z^2=a^2-x^2-y^2$$

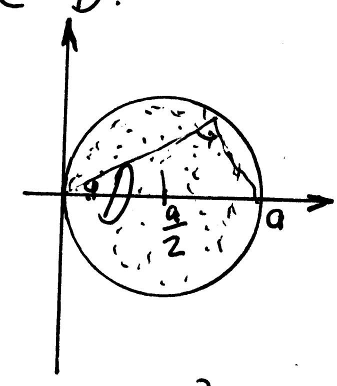
$$z=\pm\sqrt{a^2-x^2-y^2}$$

kako je u pitanju gornji dio polusfere to imamo

$$z=+\sqrt{a^2-x^2-y^2}$$

$$z'_x=\frac{-x}{\sqrt{a^2-x^2-y^2}},\quad z'_y=\frac{-y}{\sqrt{a^2-x^2-y^2}},\quad 1+z'^2_x+z'^2_y=\frac{a^2-x^2-y^2+x^2+y^2}{a^2-x^2-y^2}$$

$$P=\iint\limits_D\sqrt{\frac{a^2}{a^2-x^2-y^2}}\,dx\,dy=a\iint\limits_D\frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}$$ gdje je D:



Uvedimo polarne koordinate

$x=\rho\cos\varphi$

$y=\rho\sin\varphi$

$dx\,dy=\rho\,d\rho\,d\varphi$

$$D\xrightarrow{\text{transformiše}}D':\begin{cases}0\le\rho\le a\cos\varphi\\-\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}\end{cases}$$

$\cos\varphi=\frac{\rho}{a}$

$x^2+y^2=\ldots=\rho^2$

$$a\iint\limits_D \frac{dx\,dy}{\sqrt{a^2-(x^2+y^2)}} = \left| \begin{array}{l} \text{uvodimo} \\ \text{polarne} \\ \text{koordinate} \end{array} \right| = a\iint\limits_{D'} \frac{\rho\,d\rho\,d\varphi}{\sqrt{a^2-\rho^2}}$$

$$= a\int_{-\pi/2}^{\pi/2} d\varphi \int_0^{a\cos\varphi} \frac{\rho\,d\rho}{\sqrt{a^2-\rho^2}} = \left| \begin{array}{l} d(a^2-\rho^2) = -2\rho\,d\rho \\ \rho\,d\rho = -\frac{1}{2}\,d(a^2-\rho^2) \end{array} \right| =$$

$$= a\int_{-\pi/2}^{\pi/2} d\varphi \int_0^{a\cos\varphi} -\frac{1}{2}\,(a^2-\rho^2)^{-\frac{1}{2}}\,d(a^2-\rho^2) = -\frac{1}{2}\,2a\int_{-\pi/2}^{\pi/2} \underbrace{(a^2-\rho^2)^{\frac{1}{2}}\Big|_0^{a\cos\varphi}}_{\underbrace{(a^2-a^2\cos^2\varphi)^{\frac{1}{2}}}_{(a^2\underbrace{(1-\cos^2\varphi)}_{\sin^2\varphi})^{1/2}} - (a^2-0)^{\frac{1}{2}}} d\varphi$$

$$= -a\int_{-\pi/2}^{\pi/2} (a\sin\varphi - a)\,d\varphi$$

$$= -a^2\int_{-\pi/2}^{\pi/2} (\sin\varphi - 1)\,d\varphi = -a^2\cdot\Big(\underbrace{-\cos\varphi\Big|_{-\pi/2}^{\pi/2}}_{0} - \underbrace{\varphi\Big|_{-\pi/2}^{\pi/2}}_{\frac{\pi}{2}+\frac{\pi}{2}}\Big) = a^2\pi \quad \text{traženo rješenje}$$

(#) Pokazati da je vektorsko polje

$$\vec{v} = (2x+y+z,\ x+2y+z,\ x+y+2z)$$

potencijalno i naći njegov potencijal.

Rj.

Ako je $\operatorname{rot}\vec{v} = \vec{0}$ tada za polje $\vec{v}$ kažemo da je potencijalno polje.

$$\operatorname{rot}\vec{v} = \nabla\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+y+z & x+2y+z & x+y+2z \end{vmatrix}$$

$$= (1-1,\ -(1-1),\ 1-1) = (0,0,0)$$ vektorsko polje $\vec{v}$ je potencijalno

Potencijal polja $\vec{v}$ je f-ja u za koju vrijedi $\operatorname{grad} u = \vec{v}$.

$$\operatorname{grad} u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) = (2x+y+z,\ x+2y+z,\ x+y+2z)$$

$$\frac{\partial u}{\partial x} = 2x+y+z \quad \ldots(1)$$

$$\frac{\partial u}{\partial y} = x+2y+z \quad \ldots(2)$$

$$\frac{\partial u}{\partial z} = x+y+2z \quad \ldots(3)$$

$$(1) \Rightarrow \quad u = x^2+xy+xz+\varphi(y,z)$$

$$\frac{\partial u}{\partial y} = x+\varphi'_y \quad \ldots(4)$$

$$\frac{\partial u}{\partial z} = x+\varphi'_z \quad \ldots(5)$$

$$(2)\ i\ (4) \Rightarrow \quad \varphi'_y = 2y+z \Rightarrow \frac{\partial \varphi}{\partial y} = 2y+z \quad \ldots(6)$$

$$(3)\ i\ (5) \Rightarrow \quad \varphi'_z = y+2z \Rightarrow \frac{\partial \varphi}{\partial z} = y+2z \quad \ldots(7)$$

$$(6) \Rightarrow \quad \varphi = y^2+yz+\theta(z)$$

$$\frac{\partial \varphi}{\partial z} = y+\theta'_z \quad \ldots(8)$$

$$(7)\ i\ (8) \Rightarrow \quad \theta'_z = 2z$$

$$\theta = z^2+C$$

Sad imamo $u = x^2+xy+xz+\varphi = x^2+xy+xz+y^2+yz+\theta = x^2+y^2+z^2+xy+zx+yz+C$

Prema tome traženi potencijal je $u = x^2+y^2+z^2+xy+zx+yz+C$